Exploring Admissible Complexes of the Radon Transform over Finite Fields

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July 31, 2018

Abstract

In this paper, we consider the invertiblility of Radon transforms in the context of the finite geometries of vector spaces over finite fields. We develop the Bolker condition as a tool to verify the invertibility of Radon transform and implement it in the space of planes of the vector space $\mathbb{Z}_2^3$ over the two element field and in the space of lines of the vector space $\mathbb{Z}_2^4$. We also count the admissible complexes of planes and discuss the admissible complexes of lines, respectively, through brute force computer algorithms and also through case analysis in the context of planes in $\mathbb{Z}_2^3$ and of lines in $\mathbb{Z}_2^4$.

Summary

In this paper, we explore a specific transform, known as the Radon transform, in several contexts involving finite fields. We discuss the invertiblility of the transform in these finite geometries and try to use computer force and casework to calculate the number of collections of measurement objects (planes and lines) that give minimal invertibility for restricted Radon transforms. Such transforms are applied in the field of X-ray scanners, seismic wave detection and many other occasions as an integral part.

1 Background and Definition

This paper focuses on the Radon transform in finite geometry. We follow the theme of integral geometry as introduced by J. Radon in a 1917 paper [1], as opposed to the kind of integral geometry practiced by Buffon, Crofton, Poincaré, Chern and others. This transform and its inverse are introduced by Radon in 2-dimensional and 3-dimensional Euclidean spaces. The Radon transform is also emphasized in the work of F. John, I. M. Gelfand, S. Helgason, L. Ehrenpreis, and many others. Later researches expanded the scope of this theme. Among many others, [2] [3] also explored the Radon transform in higher dimensional Euclidean space, finite space, and non-Euclidean space. One analogous transform in the complex domain to the Radon transform is known as the Penrose transform [4]. The Radon transform is used in the recovery of data from intervals, or calculating 3D models from 2D data (known as *iterative construction*). It is known that such mathematical models could be implemented in recovering data in X-ray scanners, bar-code scanners and electron microscopy.

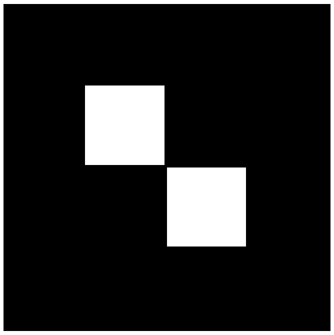
The *Radon transform* (in a two-dimensional context) is defined as the integral of the *point function* over a line. The result is a function of lines. Since our discussion will be largely in the context of finite geometry, the "integral" here indicates the summation of a function of points over a line. Similarly, we define the three-dimensional Radon transform to be the summation of a *point function* over a plane.

A **complex** Ω of objects in the n-dimensional vector space $\mathbb{Z}_q^n$ over the finite field $\mathbb{Z}_q$ is a set of geometric objects (lines, planes, etc.) such that $|\Omega| = q^n$. In other words, there are as many objects (lines, planes, etc.) in the complex as points in the vector space.

A complex Φ in a finite field $\mathbb{Z}_{||}^{\ltimes}$ is said to be *admissible* if the Radon transform over this set of objects in the vector space is injective or invertible:

$$R_\Phi : f(points\ in\ Z_q^n) \to g(\Phi)$$

A **graph** G is defined as an ordered tuple (V, E), where V is the *Vertices Set* and E is



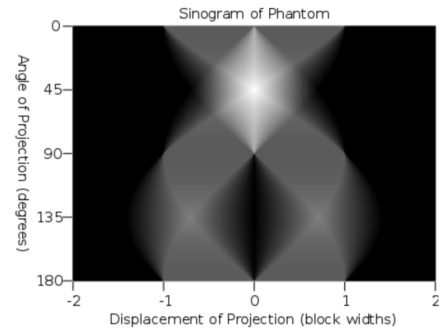


Figure 1: An illustration of a Radon transform: the left is the original function and the right is the result function

the *Edges Set*.

A **tree** $G(V, E)$ is a graph that is composed of n vertices and $n-1$ edges. In other words, $|V| = |E| + 1$. A well-known and important property of tree is that there is no cycle in a tree.

Two graphs $G_1(V_1, E_1)$ and $G_2(V_2, E_2)$ are said to have the same configurations (or equivalently, referred as *isomorphic graphs*) if and only if there exists a bijection $m : V_1 \to V_2$ such that for all points $x, y \in V_1$, the edge $xy \in E_1$ is equivalent to the edge $m(x)m(y) \in E_2$.

2 Motivation

The inverse of the Radon transform could be used in recovering a function from its integrals. In addition, it has practical applications: doctors need to recover the density distribution of biological tissue from X-ray data, which is an expression of integrals. The problem is "overdetermined" if "all" integrals (X-ray) measurements are used. Thus we are collecting a larger data set than the measurement we wish to obtain. As a result, we try to find minimal sets of data (X-rays) with which complete recovery is still possible (but possibly not stable). The *admissibility problem* asks whether a restricted Radon transform is invertible or not, and sometimes asks for the structure of all possible still-invertible restrictions.

According to a 2007 paper by Deans [5], the Radon transform could be used in picture restoration. When pictures deteriorate, the adjacent pixels are equalized and the picture becomes obscure. To reverse this process, one could apply the inverse of the Radon transform, recovering data from an"integral" (the integral of several adjacent pixels). Also, the Radon transform has practical use in determining seismic waves in the field of geophysics[6]. The wavefield could be described as $u(x, z, t)$, where x is horizontal along the earth or water surface and z is vertical. However, available seismic wave intensity sensors can only measure $u(x, t)$. In this case, the Radon transform can be used as an inverse to determine the relationship between velocity and depth of the wave.

3 Theorem for checking invertibility

Theorem 3.1 (Bolker's Theorem). *[7]*

Let a particular point and line geometry satisfy the following condition:

1. The number of lines passing through each point P is the same for all P;

2. The number of lines passing through each pair of points P, Q is the same for all pairs $P \neq Q$.

Then the Radon transform associated with this geometry is invertible with an explicit formula.

We consider the geometry of an equilateral triangle (3 points, 3 lines), a regular pentagon (5 points, 5 lines), and a regular hexagon (6 points, 6 lines) and check whether they satisfy the Bolker conditions and whether they are invertible.

Triangle: 2 lines pass through each point, 1 line passes through each pair of points. Therefore the Bolker conditions are satisfied.

Pentagon: 2 lines pass through each point. However, 0 or 1 lines pass through each pair of point. Therefore the Bolker conditions are not satisfied.

Hexagon: 2 lines pass through each point. However, 0 or 1 lines pass through each pair of point. Therefore the Bolker conditions are not satisfied.

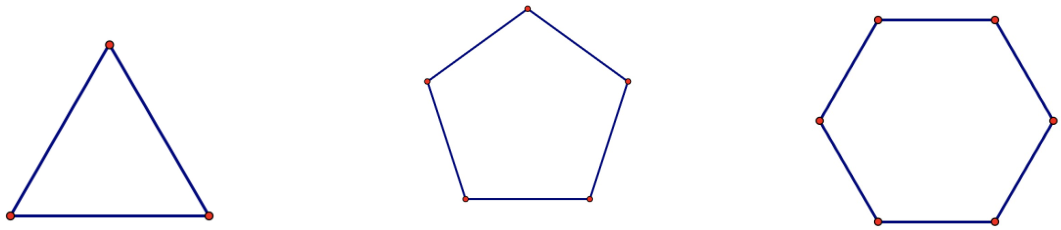


Figure 2: Simple geometries: Triangle, Pentagon, and Hexagon

To verify the invertibility of these geometries, we need to calculate matrices for them; these represent the Radon transform, and assuming that this matrix is $m \times n$, where m equals to the number of figures (in this case, lines) in a complex and n equals to the number of points. Then in row i, the figure (line) i passes through those points that have 1 as the value of their corresponding entries.

$$\textbf{Triangle}: A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

$$\textbf{Pentagon}: B = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\textbf{Hexagon}: C = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

We know that a square matrix M is invertible if and only if it is non-singular ($|M| \neq 0$). As a result, we only need to check the determinant of each matrix to verify their invertibility:

$$|A| = 2 \; |B| = 2 \; |C| = 0$$

Thus, we get the following result:

Table 1: Our results on three basic geometries

Sides	Bolker Conditions	Invertibility
3	Satisfied	Yes
5	Not Satisfied	Yes
6	Not Satisfied	No

This implies that the Bolker conditions are **sufficient** but not **necessary** conditions for invertibility of the Radon Transform.

In the next part of this paper, we discuss the plane Radon transform in the finite vector space $\mathbb{Z}_2^3$. We discuss the invertiblility of this transform and use a computer program to calculate the total number of admissible complexes.

4 Plane Radon transform in $\mathbb{Z}_2^3$

Without loss of generality, consider only the planes that pass through the origin. These must be in the form of $\alpha x + \beta y + \gamma z = 0$, where $\alpha, \beta, \gamma \in \{0, 1\}$. α, β, γ cannot be zero simultaneously, so there are $2^3 - 1 = 7$ different planes passing through the origin. Each plane is included in a parallel plane family, which is composed of 2 planes, so there are $7 \times 2 = 14$ planes in $\mathbb{Z}_2^3$.

We could verify that:

1. 7 planes pass through each point;

2. Any pair of distinct points has 3 planes passing through them;

As a result, the geometry of planes in $\mathbb{Z}_2^3$ satisfies the Bolker Conditions.

The Radon transform matrix M of this geometry is illustrated as following (14×8):

$$M = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

We could use computational brute force to determine the number of non-singular (admissible) 8×8 minors of the matrix, using a C++ program. Details of this program are included in Appendix K.1.

Our program gives a result of 648 admissible matrices (corresponding to *plane complexes* out of the total $\binom{14}{8} = 3003$ possible complexes (choose 8 columns out of 14 to get an 8×8 minor).

Now we consider the matrix $M^T M$. It is an 8×8 matrix, and could be described as the following [7]:

$$(M^T M)_{ij} = \begin{cases} \text{number of planes through a point} & i = j \\ \text{number of planes through a pair of points} & i \neq j \end{cases} \tag{1}$$

In this case, we could write out $M^T M$, using (1):

$$M^T M = \begin{bmatrix} 7 & 3 & 3 & 3 & 3 & 3 & 3 & 3 \\ 3 & 7 & 3 & 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 7 & 3 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 7 & 3 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 7 & 3 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 & 7 & 3 & 3 \\ 3 & 3 & 3 & 3 & 3 & 3 & 7 & 3 \\ 3 & 3 & 3 & 3 & 3 & 3 & 3 & 7 \end{bmatrix}$$

As a result,

$$M^T M = 4 \cdot \mathbf{I}_{8\times 8} + 3 \cdot \mathbf{1}_{8\times 8}$$

where $\mathbf{1}_{8\times 8}$ is an 8×8 matrix such that all its entries are equal to 1, and $\mathbf{I}_{8\times 8}$ is the 8×8 identity matrix.

To invert $M^T M$, we consider the following fact:

$$(a \cdot \mathbf{I} + b \cdot \mathbf{1})(c \cdot \mathbf{I} + d \cdot \mathbf{1}) = ac \cdot \mathbf{I} + (ad + bc + nbd) \cdot \mathbf{1}$$

where $n = 8$ is the size of matrix.

Thus,

$$(a \cdot \mathbf{I} + b \cdot \mathbf{1})^{-1} = \frac{1}{a} \cdot \mathbf{I} - \frac{b}{a(a + nb)} \cdot \mathbf{1}$$

Plugging in $a = 4$ and $b = 3$, we get the inverse:

$$(M^T M)^{-1} = \frac{1}{4} \cdot \mathbf{I} - \frac{3}{112} \cdot \mathbf{1}$$

In the next part of this paper, we will attempt to count the number of admissible line complexes in $\mathbb{Z}_2^4$.

In $\mathbb{Z}_2^4$, there are 16 points and $\binom{16}{2} = 120$ lines. 15 lines pass through each point and 1 line passes through each pair of points. So the Bolker conditions are satisfied.

Every line complex contains 16 lines, and there are

$$\binom{120}{16} = 31,044,058,215,401,404,845 \approx 3.10 \times 10^{19}$$

line complexes in $\mathbb{Z}_2^4$.

As mentioned in [7] [8], a line complex Φ is admissible if and only if the following three conditions hold:

1. Φ omits no points;

2. Φ has no isolated subtrees; (Note that a point is not considered as an isolated tree)

3. Φ contains no even cycles.

Now we attempt to count the number of admissible line complexes in $\mathbb{Z}_2^4$ by trying to count some cases that contradict one or more of the three conditions above, by using the Inclusion-Exclusion Principle and casework.

5 Line complexes that omit points

If a line complex omits 10 points, then the 6 points remaining could only form $\binom{6}{2} = 15 < 16$ lines, which cannot form a line complex. So a line complex can omit at most 9 points (since $\binom{7}{2} = 21 > 16$).

5.1 Line complexes that omit at least one point

To choose the omitted point, there are $\binom{16}{1}$ possibilities; the 16 lines must be chosen among the lines that are formed by the remaining 15 points, of which there are $\binom{15}{2} = 105$ in total. Hence, the total number of complexes omitting a point is $\binom{16}{1} \cdot \binom{105}{16}$. However, we must note that each complex in this case is counted with multiplicity equal to the number of points that it misses. This number can, of course, be greater than 1.

5.2 Line complexes that omit at least two points

In order to eliminate the multiplicity mentioned above, we attempt to count line complexes that omit more than one point. To choose the omitted pair of points, there are $\binom{16}{2}$ possibilities; the 16 lines must be chosen among the lines that are formed by the remaining 14 points. Therefore there are $\binom{14}{2} = 91$ possibilities in total. Hence, the total number of complexes in this case is $\binom{16}{2} \cdot \binom{91}{16}$.

5.3 Line complexes that omit at least three or more points

Applying similar analysis as above, we could calculate the number of complexes that at least three or more points are omitted:

At least 3 points: $\binom{16}{3} \cdot \binom{78}{16}$

At least 4 points: $\binom{16}{4} \cdot \binom{66}{16}$

At least 5 points: $\binom{16}{5} \cdot \binom{55}{16}$

At least 6 points: $\binom{16}{6} \cdot \binom{45}{16}$

At least 7 points: $\binom{16}{7} \cdot \binom{36}{16}$

At least 8 points: $\binom{16}{8} \cdot \binom{28}{16}$

9 points: $\binom{16}{9} \cdot \binom{21}{16}$

Recall that a line complex cannot omit more than 9 points, so this list is exhaustive.

5.4 Summation of the number of line complexes that omit at least one point

Using the Inclusion-Exclusion Principle, we could obtain the following result for the number of line complexes that omit at least one point:

$$\begin{aligned}&\binom{16}{1}\cdot\binom{105}{16}-\binom{16}{2}\cdot\binom{91}{16}+\binom{16}{3}\cdot\binom{78}{16}-\binom{16}{4}\cdot\binom{66}{16}+\binom{16}{5}\cdot\binom{55}{16}\\&-\binom{16}{6}\cdot\binom{45}{16}+\binom{16}{7}\cdot\binom{36}{16}-\binom{16}{8}\cdot\binom{28}{16}+\binom{16}{9}\cdot\binom{21}{16}\\&=27,038,432,931,510,128,310\approx 2.70\times 10^{19}.\end{aligned}$$

6 Line complexes that contain isolated lines

If a line complex has 6 isolated lines, then the 10 remaining lines could only be formed from $16-2\times 6=4$ points, which is impossible. So a line complex can contain at most 5 isolated lines (It is possible to form the remaining 11 lines from $16-2\times 5=6$ points).

6.1 Line complexes that contain at least one isolated line

To choose the isolated line, there are $\binom{16}{2}=120$ possibilities; the number of lines that do not intersect this isolated line is $\binom{14}{2}=91$. The remaining 15 lines must be chosen from these 91 lines, so the number of such line complexes is: $120\cdot\binom{91}{15}$.

6.2 Line complexes that contain at least two isolated lines

To choose the two isolated lines, there are $\binom{16}{2}=120$ possibilities for the first line; since the number of lines that do not intersect this isolated line is $\binom{14}{2}=91$, there are 91 choices for the second line. Note that the first and second isolated line are interchangeable, so the total number of ways to to choose them is $\dfrac{120\cdot 91}{2}$. The remaining 14 lines must be chosen from the $\binom{12}{2} = 66$ lines that do not intersect them, so the total number of possibilities is: $\frac{120 \cdot 91}{2} \cdot \binom{66}{14}$.

6.3 Line complexes that contain at least three or more isolated lines

The case where there are at least 3, 4 and 5 isolated lines can be found in Appendix A. Recall that a line complex cannot have more than 5 isolated lines.

6.4 Summation of the number of line complexes that contain at least one isolated line

Using the Inclusion-Exclusion Principle, we obtain the following result:

$$\begin{aligned}
&120 \cdot \binom{91}{15} - \frac{120 \cdot 91}{2} \cdot \binom{66}{14} + \frac{120 \cdot 91 \cdot 66}{6} \cdot \binom{45}{13} \\
&- \frac{120 \cdot 91 \cdot 66 \cdot 45}{24} \cdot \binom{28}{12} + \frac{120 \cdot 91 \cdot 66 \cdot 45 \cdot 28}{120} \cdot \binom{15}{11} \\
&= 6,166,165,232,542,385,070 \approx 6.17 \times 10^{18}.
\end{aligned}$$

7 Line complexes that contain both omitted points and isolated lines

We apply casework based on the number of omitted points in the line complex. Let C be a line complex. Let $m(C)$ denote the number of (point, line) pairs (p, ℓ), so that C omits p and has ℓ as an isolated line. Clearly, $m(C) = |p| \cdot |\ell|$ (according to the Multiplication Principle), by which we mean the number of point-line pairs. This definition would be used for summation in later part of this section.

7.1 Line complexes that contain at least one omitted point and at least one isolated line

7.1.1 At least 1 isolated line

Here, $m(C) \geq 1$. There are $\binom{16}{1}$ choices for the point, and $\binom{15}{2} = 105$ ways to choose the line. The remaining 13 points can form $\binom{13}{2} = 78$ lines. Hence, the total number of approaches in this case is: $\binom{16}{1} \cdot 105 \cdot \binom{78}{15}$.

7.1.2 At least 2 isolated lines

Here, $m(C) \geq 2$. There are $\binom{16}{1}$ methods to choose the point, $\binom{15}{2} = 105$ methods to choose the first line, and $\binom{13}{2} = 78$ methods to choose the second line. We must divide it by two since the first and second isolated line are interchangeable. The remaining 11 points can form $\binom{11}{2} = 55$ lines. Hence, the total number of complexes in this case is: $\binom{16}{1} \cdot \dfrac{105 \cdot 78}{2} \cdot \binom{55}{14}$.

7.1.3 At least 3, 4 isolated lines

These cases are included in Appendix B.

It is impossible to have 5 or more isolated line while an omitted point is present. If there are 5 isolated lines, then the point-line group takes up $1 + 2 \times 5 = 11$ points, and the remaining 5 points could only form $\binom{5}{2} = 10$ lines, not enough to create a line complex.

7.2 Line complexes that contain at least two omitted points

7.2.1 At least 1 isolated line

Here, $m(C) \geq 2$. To save space, we just give the result: $\binom{16}{2} \cdot 91 \cdot \binom{66}{15}$.

7.2.2 At least 2 isolated lines

Here, $m(C) \geq 4$. The enumeration for this case is: $\binom{16}{2} \cdot \dfrac{91 \cdot 66}{2} \cdot \binom{45}{14}$.

7.2.3 At least 3, 4 isolated lines

These cases are included in Appendix C.

It is impossible to have more than 4 isolated lines in this case (when there are 5 isolated lines, the point-line group takes up $2 + 5 \times 2 = 12$ points, and the remaining 4 points could only form $\binom{4}{2} = 6$ lines, which is not enough to create a line complex).

7.3 Line complexes that contain at least 3-8 omitted points

These cases are included in Appendixes D-I.

7.4 Line complexes that contain at least nine omitted points

In fact, it is impossible to have more than eight omitted points and at least one isolated line. If there are 9 omitted points and one isolated line, then they take up $9 + 1 \times 2 = 11$ points, and the remaining 5 points could only form $\binom{5}{2} = 10$ lines, which is not enough to create a line complex.

7.5 Summation of the number of line complexes that contain both omitted point and isolated line

Now let M_i denote the number of line complexes C with $m(C) \geq i$. Applying the Inclusion-Exclusion Principle, we need to compute

$$M_1 - M_2 + M_3 - M_4 + \ldots + (-1)^{k+1} M_k \ldots .$$

In other words, when i is even, M_i will have negative sign in the summation; when i is odd, M_i will have positive sign.

Using the above equation, the total number of line complexes that contain both omitted point and isolated line is: $4,539,727,750,144,346,760 \approx 4.54 \times 10^{18}$.

8 Complexes with isolated trees

It is known that the number of trees that have n vertices and different configurations (topological structure) is the $(n-1)$st Catalan number, or $h_{n-1} = \frac{\binom{2n-2}{n-1}}{n}$ (recall the definition of graph, tree, and graphs that have the same configuration in Section 1). The largest isolated trees in a 16-point graph could contain 8 vertices, which has $h_7 = \frac{\binom{14}{7}}{8} = 429$ configurations. Since the number of cases is huge, we will only discuss the simplest cases in this paper.

8.1 Complexes with "L-shaped" isolated trees

Consider those complexes with 3-point trees, where one point has a degree of 2 and the remaining two points have a degree of 1.

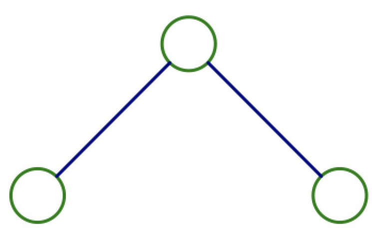


Figure 3: An L-shaped tree. The vertex in the middle has a degree of 2, while the remaining vertices have a degree of 1 respectively.

There are $\binom{16}{3}$ ways to choose the three vertices of the L-shaped tree, and 3 ways to choose the vertex that has a degree of 2. For the remaining 14 lines, they must be chosen from the lines that are formed from the remaining 13 points, which has a total number of $\binom{\binom{13}{2}}{14} = \binom{78}{14}$ ways. Hence the total number of complexes with L-shaped trees is: $\binom{16}{3} \cdot 3 \cdot \binom{78}{14} \approx 1.72 \times 10^{18}$

However, we must notice that this is only a rough upper bound of the actual result, since we have included some cases where there are isolated points. Furthermore, some complexes in which there are more than one L-shaped trees (up to 4) are counted multiple times.

To illustrate with more detail, the following cases must be examined:

1. Complexes with at least one L-shaped isolated tree and omitted point.
2. Complexes with two L-shaped isolated trees.
3. Complexes with three L-shaped isolated trees.

Note that there are also some complexes that are both in case 1 and case 2 or 3. There do not exist complexes with four or more L-shaped isolated trees, since there are $2 \times 4 = 8$ lines in the trees, the remaining $16 - 3 \times 4 = 4$ points could form at most $\binom{4}{2} = 6$ lines. A total of $8 + 6 = 14$ lines is not enough to form a complex.

To provide readers with a rough sense of how many complexes that contain at least one L-shaped tree fall in case 1, we use a program to choose 100 million complexes that have at least one L-shaped tree randomly and examine whether they omit points. Details of this program are included in Appendix K.2.e

The program gives a result of $ans = 6,942,230$, so we could deduce that there are approximately 6.94% complexes containing a L-shaped tree that do not omit points. Thus, there are approximately $1.72 \times 10^{18} \times 6.94\% = 1.19 \times 10^{17}$ complexes that contain at least one L-shaped tree and omit no point.

8.2 Complexes with "T-shaped" isolated trees

This case is mainly discussed in Appendix J.

9 Complexes containing cycles

Here we will consider line complexes with one or more even cycles. Since a complex may contain multiple cycles, and some of the cases may be very complicated (e.g. a complex containing a 10-cycle and a 4-cycle), we will just focus on some relatively simple cases.

9.1 Complexes containing a 16-cycle

Denote the 16 points in a complex $p_1, p_2, \ldots, p_{16}$. In this case, all lines in the complex form the cycle. To count the number of complexes which satisfy this condition, we fix a line $\ell = p_1 p_2$. Then we order the remaining 14 points to form the cycle. We have $\binom{16}{2} = 120$ ways to choose the fixed line and 14! ways to order the 14 points. However, our fixed line could be any of the 16 lines. As a result, the total number of complex is: $\dfrac{120 \cdot 14!}{16} = 653837184000 \approx 6.54 \times 10^{11}$

9.2 Complexes containing a 14-cycle

To begin with, note that it is impossible to have more than one 14-cycle in a complex. If there exists a second 14-cycle in a complex, then the remaining two lines (14 lines have been used by the first cycle) must be used to "reach out" to the remaining two points. However, there are no lines left to connect the two points.

To calculate the total number of complexes which contain a 14-cycle, we select 14 points out of 16 to form the cycle, which has $\binom{16}{14}$ ways. Then, we order the 14 points and by eliminating the 14 possible rotations and 2 possible orientations, we get a total of $\dfrac{14!}{14 \cdot 2}$ approaches.

There are three possible ways to attach the remaining 2 points A, B (shown in Figure 4 respectively).

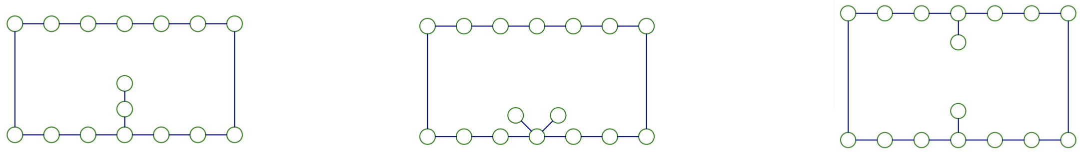


Figure 4: Three different cases of line complexes containing a 14-cycle respectively

In the first case, we use one line to connect the cycle with one point A and then connect the other point B with A. In this case, we choose the point in the cycle that A is attached to, and then choose A out of A and B. Thus, the number of approaches in this case is $14 \times 2 = 28$.

In the second case, both A and B are attached to one point in the cycle. We only need to choose this point of attachment, which has 14 different possible approaches.

In the third case, A and B are attached to different points in the cycle. We choose the two points from 14 points in the cycle and then designate which point A and B is attached to. There are $\binom{14}{2} \times 2 = 182$ approaches in this case.

The total number of complex which contains a 14-cycle is: $\dfrac{14!}{14 \cdot 2} \cdot (28 + 14 + 182) = 697,426,329,600 \approx 6.97 \times 10^{11}$

10 Conclusion

In this paper, we have presented the Radon transform, explored its invertiblility and managed to calculate the number of admissible complexes for planes in $\mathbb{Z}_2^3$ and discussed the number of admissible complexes for lines in $\mathbb{Z}_2^4$.

11 Acknowledgments

I would like to thank my mentor, Prof. Eric Grinberg, and all people who gave me invaluable guidance whenever I am faced with obstacles that I could not solve by myself.

References

[1] J. Radon. Über die bestimmung von funktionen durch ihre integralwerte längs gewisser mannigfaltigkeiten. *Leipzig:Math.-Phys. Kl.*, 69:262–267, 1917.

[2] L. A. Kunyansky. Explicit inversion formulae for the spherical mean radon transform. *Inverse problems*, 23(1):373, 2007.

[3] O. Tretiak and C. Metz. The exponential radon transform. *SIAM Journal on Applied Mathematics*, 39(2):341–354, 1980.

[4] R. J. Baston and M. G. Eastwood. *The Penrose transform: its interaction with representation theory*. Courier Dover Publications, 2016.

[5] S. R. Deans. *The Radon transform and some of its applications*. Courier Corporation, 2007.

[6] W. Moon, A. Carswell, R. Tang, and C. Dilliston. Radon transform wave field separation for vertical seismic profiling data. *Geophysics*, 51(4):940–947, 1986.

[7] E. L. Grinberg. The admissibility theorem for the spatial x-ray transform over the two-element field. pages 111–123, 2012.

[8] E. L. Grinberg and M. Orhon. A scrapbook of inadmissible line complexes for the spatial x-ray transform over the two-element field.

A More cases for complexes that have isolated lines

Applying similar analysis as above, we could get the result for line complexes that contain more than two isolated lines:

At least 3 isolated lines:

$$\frac{120 \cdot 91 \cdot 66}{6} \cdot \binom{45}{13}$$

At least 4 isolated lines:

$$\frac{120 \cdot 91 \cdot 66 \cdot 45}{24} \cdot \binom{28}{12}$$

5 isolated lines:

$$\frac{120 \cdot 91 \cdot 66 \cdot 45 \cdot 28}{120} \cdot \binom{15}{11}$$

B More cases for complexes that omit at least one point

At least 3 isolated lines

Here, $m(C) \geq 3$. Using the similar analysis as above, we get:

$$\binom{16}{1} \cdot \frac{105 \cdot 78 \cdot 55}{6} \cdot \binom{36}{13}$$

At least 4 isolated lines

Here, $m(C) \geq 4$. Using the similar analysis as above, we get:

$$\binom{16}{1} \cdot \frac{105 \cdot 78 \cdot 55 \cdot 36}{24} \cdot \binom{21}{12}$$

C More cases for complexes that omit at least two points

At least 3 isolated lines

$m(C) \geq 6$.

$$\binom{16}{2} \cdot \frac{91 \cdot 66 \cdot 45}{6} \cdot \binom{28}{13}$$

At least 4 isolated lines

$m(C) \geq 8$.

$$\binom{16}{2} \cdot \frac{91 \cdot 66 \cdot 45 \cdot 28}{24} \cdot \binom{15}{12}$$

D Line complexes that contain at least three omitted points

D.1 At least 1 isolated line

$m(C) \geq 3$.

$$\binom{16}{3} \cdot 78 \cdot \binom{55}{15}$$

D.2 At least 2 isolated lines

$m(C) \geq 6$.

$$\binom{16}{3} \cdot \frac{78 \cdot 55}{2} \cdot \binom{36}{14}$$

D.3 At least 3 isolated lines

$m(C) \geq 9$.

$$\binom{16}{3} \cdot \frac{78 \cdot 55 \cdot 36}{6} \cdot \binom{21}{13}$$

It is impossible to have more than 3 isolated lines in this case (when there are 4 isolated lines, the point-line group takes up $3 + 4 \times 2 = 11$ points, and the remaining 45 points could only form $\binom{4}{2} = 10$ lines, not enough to create a line complex).

E Line complexes that contain at least four omitted points

E.1 At least 1 isolated line

$m(C) \geq 4$.

$$\binom{16}{4} \cdot 66 \cdot \binom{45}{15}$$

E.2 At least 2 isolated lines

$m(C) \geq 8$.

$$\binom{16}{4} \cdot \frac{66 \cdot 45}{2} \cdot \binom{28}{14}$$

E.3 At least 3 isolated lines

$m(C) \geq 12$.

$$\binom{16}{4} \cdot \frac{66 \cdot 45 \cdot 28}{6} \cdot \binom{15}{13}$$

It is impossible to have more than 3 isolated lines in this case.

F Line complexes that contain at least five omitted points

F.1 At least 1 isolated line

$m(C) \geq 5$.

$$\binom{16}{5} \cdot 55 \cdot \binom{36}{15}$$

F.2 At least 2 isolated lines

$m(C) \geq 10$.

$$\binom{16}{5} \cdot \frac{55 \cdot 36}{2} \cdot \binom{21}{14}$$

It is impossible to have more than 2 isolated lines in this case.

G Line complexes that contain at least six omitted points

G.1 At least 1 isolated line

$m(C) \geq 6$.

$$\binom{16}{6} \cdot 45 \cdot \binom{28}{15}$$

G.2 At least 2 isolated lines

$m(C) \geq 12$.

$$\binom{16}{6} \cdot \frac{45 \cdot 28}{2} \cdot \binom{15}{14}$$

It is impossible to have more than 2 isolated lines in this case.

H Line complexes that contain at least seven omitted points

H.1 At least 1 isolated line

$m(C) \geq 7$.

$$\binom{16}{7} \cdot 36 \cdot \binom{21}{15}$$

It is impossible to have more than 1 isolated lines in this case.

I Line complexes that contain at least eight omitted points

I.1 At least 1 isolated line

$m(C) \geq 8$.

$$\binom{16}{8} \cdot 28 \cdot \binom{25}{15}$$

It is impossible to have more than 1 isolated lines in this case.

J Complexes with "T-shaped" isolated trees

Consider those complexes with 4-point trees, where one point has a degree of 3 and the remaining three points has a degree of 1.

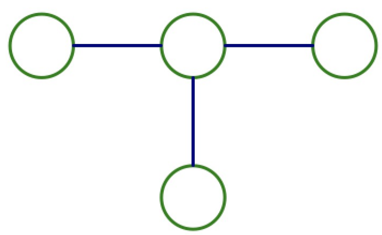


Figure 5: T-shaped tree

Applying a similar analysis as in the L-shaped tree case, the total number of complexes with T-shaped trees is: $\binom{16}{4} \cdot 4 \cdot \binom{66}{13} \approx 1.489 \times 10^{17}$

However, the following case must be examined to avoid multiplicity:

1. Complexes with at least one T-shaped isolated tree and one omitted point.

2. Complexes with two T-shaped isolated trees. (Case 1 and 2 have an overlapping part)

There do not exist complexes with three or more T-shaped isolated trees, since there are $3 \times 3 = 9$ lines in the trees, the remaining $16 - 4 \times 3 = 4$ points could form at most $\binom{4}{2} = 6$

lines. And a total of $9 + 6 = 15$ lines is not enough to form a complex.

K Programs

K.1 Program for calculating the number of admissible plane complexes in $\mathbb{Z}_2^3$

```
int a = 0, cur = 0;
int mat[14][8]; // let mat = M (shown above)
for(int a1 = 0; a1 <= 6; a1++)
  for(int a2 = a1 + 1; a2 <= 7; a2++)
    for(int a3 = a2 + 1; a3 <= 8; a3++)
      for(int a4 = a3 + 1; a4 <= 9; a4++)
        for(int a5 = a4 + 1; a5 <= 10; a5++)
          for(int a6 = a5 + 1; a6 <= 11; a6++)
            for(int a7 = a6 + 1; a7 <= 12; a7++)
              for(int a8 = a7 + 1; a8 <= 13; a8++)
                int minor[8][8]; // a 8 * 8 minor of mat
                minor[0] = mat[a1];
                minor[1] = mat[a2];
                minor[2] = mat[a3];
                minor[3] = mat[a4];
                minor[4] = mat[a5];
                minor[5] = mat[a6];
                minor[6] = mat[a7];
                minor[7] = mat[a8];
                a = det (minor) /* det(minor)
                is the determinant of minor)*/
                if(a != 0)
                  printf ("%d,%d,%d,%d,%d,%d,%d,%d",
                  a1, a2, a3, a4, a5, a6, a7, a8);
                  cur++;
```

K.2 Program for approximating the number of line complexes containing at least one L-shaped tree and not omitting any point in $\mathbb{Z}_2^4$

```
int tree[125][2];
bool used[20];
int cnt = 0, ans = 0;
for(int i = 1; i <= 15; i++)
        for(int j = i + 1; j <= 16; j++)
        {
                cnt++;
                tree[cnt][0] = i;
                tree[cnt][1] = j;
        } // Initializing the tree
for(int x = 1; x <= 100000000; x++)
{
        memset(used, 0, 20 * sizeof(bool));
        used[1] = used[2] = used[3] = 1; /* Assigning the L-shaped tree.
        The edges are 1-2 and 2-3 */
        for(int p = 1; p <= 13; p++)
        {
                srand(x * p);
                int s = random(119) + 1;
                used[tree[s][0]] = 1;
                used[tree[s][1]] = 1; // Randomly choosing the remaining 13 edges
        }
        int res = 0;
        for(int q = 1; q <= 16; q++)
        {
                if(used[q] == 1)
                        res++; // res is the number of vertices that are "used" by the tree
        }
        if(res == 16)
                ans++; // ans is the number of trees that do not omit any point
}
cout << ans << endl;
```